HW IV

Let $x_n \in \mathbb{R}\ \forall n$ and $S_n := x_1 + \cdots + x_n$; let $r \in (0,1)$.

1. Show $\lim_N \sum_{n=1}^{N} r^n$ exists (in $\mathbb{R}$) for each of the following methods:
   (i) use the Bounded Monotone Conv. Th.
   (ii) $\left(\sum_{n=1}^{N} r^n : N \in \mathbb{N}\right)$ is a Cauchy seq.

2. Let (x_n) be a contractive seq with rate r:
$$|x_{n+1} - x_n| \leqslant r\,|x_n - x_{n-1}| \quad \forall n \in \mathbb{N},\ n > 1. \qquad (1)$$
Show that
$$|x_{n+1} - x_n| \leqslant r^{n-1}\,|x_2 - x_1| \quad \forall n \in \mathbb{N} \qquad (2)$$
and that
$$|x_{n+k} - x_n| \leqslant \frac{r^{n-1}}{1-r}\,|x_2 - x_1| \quad \forall n, k \in \mathbb{N}. \qquad (3)$$
Using <u>each</u> of the following suggestions, show that (x_n) converges.
   (i) $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ is absolutely convergent and so convergent (hence $\lim_n (x_{n+1} - x_1)$ exists in $\mathbb{R}$); $\therefore \lim_n x_n$ exists
   (ii) By (3), (x_n) is Cauchy.

3. Let $x_1 := 1$, $x_2 := 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})\ \forall n > 2$. Show by Q2 that (x_n) converges. But it is quite difficult to find its limit: show by MI that
$$x_{2n-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{2n-3} \longrightarrow 1 + \frac{1}{2}\left(\frac{1}{1-1/4}\right) = 5/3$$
$$x_{2n} = 2 - \frac{1}{2^2} - \frac{1}{2^4} - \cdots - \left(\frac{1}{2}\right)^{2n-2} \longrightarrow 2 - \frac{1}{4}\left(\frac{1}{1-1/4}\right) = 5/3$$
Hence $\lim_n x_n = 5/3$ (Why?)

4. Let $0 \leqslant x_n \downarrow_n$ (i.e. $x_n \geqslant x_{n+1}\ \forall n$) and let $t_n := \sum_{k=1}^{n} 2^k x_{2^k}$. Then, in "grouping",
$$x_1 + (x_2 + x_3) + (x_4 + \cdots + x_{2^3-1}) + (x_{2^3} + \cdots + x_{2^4-1}) + \cdots + (x_{2^{n-1}} + \cdots + x_{2^n-1})$$
$$\leqslant x_1 + (x_2 \times 2) + (x_{2^2} \times 2^2) + (x_{2^3} \times 2^3) + \cdots (x_{2^{n-1}} \times 2^{n-1}) = \sum_{k=1}^{n} x_{2^{k-1}} \cdot 2^{k-1}. \qquad (1)$$
Also, in different grouping,
$$x_1 + x_2 + (x_3 + x_4) + (x_5 + \cdots + x_8) + \cdots + (x_{2^{n-1}+1} + \cdots x_{2^n})$$
$$\geqslant x_1 + x_2 + 2 \cdot x_{2^2} + 2^2 x_{2^3} + \cdots + 2^{n-1} x_{2^n} \geqslant x_1 + 2^{-1}\left(\sum_{k=1}^{n} x_{2^k} \cdot 2^k\right) \qquad (2)$$

4 (continue). Thus $s_{2^n-1} \le t_n$ and $s_{2^n} \ge x_1 + \frac{1}{2} t_n$ $\forall n \ge 2$ and so $\sum_{n=1}^{\infty} x_n$ conv iff its "condensation" series $\sum_{n=1}^{\infty} 2^n x_{2^n}$ conv. (Cauchy condensation theorem).

Remark. The method of different ways of grouping is also used to deal with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (div), and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (conv) for $p > 1$.
(Can you do them?)

5. Let $a > 0$ and $z_1 > 0$. Let ξ, M be defined by
$\xi := \max\{1, z_1\}$ and $M = \sqrt{a} + \xi$
(so $\sqrt{a+ [illegible]} \le \sqrt{a} + [illegible]$). Let (z_n) be defined by
$$z_n = \sqrt{a + z_{n-1}} \quad \forall n > 1.$$
Show, by MI, that each $z_n \le M$ and (z_n) $\uparrow$ or $\downarrow$ (depending on $z_1 \le z_2$ or $z_1 \ge z_2$). Why the seq converges and to what (which root of an equation)?

6*. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2}(x_n + 3)$ $\forall n$. Using each of the following suggestions to show the convergence of (x_n) [and find the limit]:
(i) (x_n) is $\uparrow$ and bounded by 3
(ii) (x_n) is contractive: $|x_{n+2} - x_{n+1}| \le r|x_{n+1} - x_n|$ with appropriate $r \in (0,1)$.

7*. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above with $s := \sup A$. Suppose $s \notin A$. Show that $\exists$ (a_n) strictly increasing such that $\lim_n a_n = s$.

8*. Q10 of p74. (regarding limit superior/inferior).